Lecture 18: Cohomology and Universal Coefficient Theorem

Cohomology

R refers to a commutative ring in this section.

Definition

A cochain complex over R is a sequence of R-module maps

$$\cdots \to C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n-1} \to \cdots$$

such that $d_n \circ d_{n-1} = 0$. When R is not specified, we mean cochain complex of abelian groups (i.e. $R = \mathbb{Z}$).

Sometimes we just write the cochain complex by $(C^\bullet, d)$. Then

$$d_n = d|_{C_n} \quad \text{and} \quad d^2 = 0.$$

Definition

Given a cochain complex $(C^\bullet, d)$, its n-cocycles Z^n and n-coboundaries B^n are

$$Z^n = \mathrm{Ker}(d : C^n \to C^{n+1}), \quad B^n = \mathrm{Im}(d : C^{n-1} \to C^n).$$

$d^2 = 0$ implies $B^n \subset Z^n$. We define the n-th cohomology group by

$$\mathrm{H}^n(C^\bullet, d) := \frac{Z^n}{B^n} = \frac{\ker(d_n)}{\mathrm{im}(d_{n-1})}.$$

A cochain complex $C^\bullet$ is called acyclic or exact if

$$\mathrm{H}^n(C^\bullet) = 0 \quad \text{for all } n.$$

Definition

Let $(C_\bullet, \partial)$ be a chain complex over R, and G be a R-module. We define its dual cochain complex $(C^\bullet, d) = \mathrm{Hom}_R(C_\bullet, G)$ by

$$\cdots \mathrm{Hom}_R(C_{n-1}, G) \to \mathrm{Hom}_R(C_n, G) \to \mathrm{Hom}_R(C_{n+1}, G) \to \cdots$$

Here given $f \in \mathrm{Hom}_R(C_n, G)$, we define

$$d_n f \in \mathrm{Hom}_R(C_{n+1}, G)$$

by

$$d_n f(c) := f(\partial_{n+1}(c)), \quad \forall c \in C_{n+1}.$$

Definition

Let G be an abelian group and X be a topological space. For $n \geq 0$, we define the group of singular n-cochains in X with coefficient in G to be

$$S^n(X; G) := \mathrm{Hom}(S_n(X), G).$$

The dual cochain complex $S^\bullet(X; G) = \mathrm{Hom}(S_\bullet(X), G)$ is called the singular cochain complex with coefficient in G. Its cohomology is called the singular cohomology and denoted by

$$\mathrm{H}^n(X; G) := \mathrm{H}^n(S^\bullet(X; G)).$$

When $G = \mathbb{Z}$, we simply write it as $\mathrm{H}^n(X)$.

We have the analogue of chain homotopy between cochain complexes.

Theorem

$\mathrm{H}^n(-;G)$ defines a contra-variant functor

$$\mathrm{H}^n(-;G) : \mathbf{hTop} \to \mathbf{Ab}\,.$$

Theorem (Dimension Axiom)

If X is contractible, then

$$\mathrm{H}^n(X;G) = \begin{cases} G & n = 0 \\ 0 & n > 0 \end{cases}$$

Lemma

Let G be a R-module and $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of R-modules. Then the following sequence is exact

$$0 \to \mathrm{Hom}_R(A_3, G) \to \mathrm{Hom}_R(A_2, G) \to \mathrm{Hom}_R(A_1, G).$$

If A_3 is a free R-module (or more generally projective R-module), then the last morphism is also surjective.

Definition

Let G be an abelian group. Let $A \subset X$ be a subspace. We define the relative singular cochain complex with coefficient in G by

$$S^\bullet(X, A; G) := \mathrm{Hom}(S_\bullet(X)/S_\bullet(A), G).$$

Its cohomology $\mathrm{H}^\bullet(X, A; G)$ is called the relative singular cohomology.

Since $S_\bullet(X)/S_\bullet(A)$ is a free abelian group, we have a short exact sequence of cochain complex

$$0 \to S^\bullet(X, A; G) \to S^\bullet(X; G) \to S^\bullet(A; G) \to 0$$

which induces a long exact sequence of cohomology groups

$$0 \to \mathrm{H}^0(X, A; G) \to \mathrm{H}^0(X; G) \to \mathrm{H}^0(A; G) \to \mathrm{H}^1(X, A; G) \to \cdots .$$

Moreover, the connecting maps

$$\delta : \mathrm{H}^n(A, G) \to \mathrm{H}^{n+1}(X, A; G)$$

is natural in the same sense as that for homology.

Theorem (Excision)

Let $U \subset A \subset X$ be subspaces such that $\bar{U} \subset A^\circ$ (the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms

$$i^* : \mathrm{H}^n(X, A; G) \simeq \mathrm{H}^n(X - U, A - U; G), \quad \forall n.$$

Theorem (Mayer-Vietoris)

Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence

$$\cdots \to \mathrm{H}^n(X; G) \to \mathrm{H}^n(X_1; G) \oplus \mathrm{H}^n(X_2; G) \to \mathrm{H}^n(X_1 \cap X_2; G) \to \mathrm{H}^{n+1}(X; G) \to \cdots$$

Universal Coefficient Theorem for Cohomology

Definition

Let M, N be two R-modules. Let $P_\bullet \to M$ be a free R-module resolution of M:

$$\cdots P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0$$

is an exact sequence of R-modules and P_i's are free.

We define the Ext group

$$\mathrm{Ext}^k_R(M, N) = \mathrm{H}^k(\mathrm{Hom}(P_\bullet, N))$$

and the Tor group

$$\mathrm{Tor}^R_k(M, N) = \mathrm{H}_k(P_\bullet \otimes_R N).$$

Note that

$$\operatorname{Ext}^0_R(M, N) = \operatorname{Hom}_R(M, N), \quad \operatorname{Tor}^R_0(M, N) = M \otimes_R N.$$

Ext and Tor are called the derived functors of Hom and $\otimes$.

It is a classical result in homological algebra that $\operatorname{Ext}^k_R(M, N)$ and $\operatorname{Tor}^R_k(M, N)$ do not depend on the choice of resolutions of M. They are functorial with respect to both variables and Tor^R_k is symmetric in two variables

$$\operatorname{Tor}^R_k(M, N) = \operatorname{Tor}^R_k(N, M).$$

Moreover, for any short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

there are associated long exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N) \\ &\to \mathrm{Ext}^1_R(M_3, N) \to \mathrm{Ext}^1_R(M_2, N) \to \mathrm{Ext}^1_R(M_1, N) \\ &\to \mathrm{Ext}^2_R(M_3, N)) \to \mathrm{Ext}^2_R(M_2, N) \to \mathrm{Ext}^2_R(M_1, N) \to \cdots \end{aligned}$$

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_3) \\ &\to \mathrm{Ext}^1_R(N, M_1) \to \mathrm{Ext}^1_R(N, M_2) \to \mathrm{Ext}^1_R(N, M_3) \\ &\to \mathrm{Ext}^2_R(N, M_1)) \to \mathrm{Ext}^2_R(N, M_2) \to \mathrm{Ext}^2_R(N, M_3) \to \cdots \end{aligned}$$

and

$$\begin{aligned}\cdots &\to \mathrm{Tor}_2^R(M_1, N) \to \mathrm{Tor}_2^R(M_2, N) \to \mathrm{Tor}_3^R(M_3, N)\\ &\to \mathrm{Tor}_1^R(M_1, N) \to \mathrm{Tor}_1^R(M_2, N) \to \mathrm{Tor}_1^R(M_3, N)\\ &\to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0\end{aligned}$$

Now we focus on the case of abelian groups $R=\mathbb{Z}$. For any abelian group M, let P_0 be a free abelian group such that $P_0\to M$ is surjective. Let P_1 be its kernel. Then P_1 is also free and

$$0\to P_1\to P_0\to M\to 0$$

defines a free resolution of abelian groups. This implies that

$$\mathrm{Ext}^k(M,N)=0,\quad \mathrm{Tor}_k(M,N)=0 \quad \text{for} \quad k\geq 2.$$

For abelian groups we will simply write

$$\boxed{\mathrm{Ext}(M, N) := \mathrm{Ext}^1_{\mathbb{Z}}(M.N), \quad \mathrm{Tor}(M, N) := \mathrm{Tor}^{\mathbb{Z}}_1(M, N)}.$$

Lemma

If either M is free or N is divisible, then $\mathrm{Ext}(M, N) = 0$.

Proposition

Let $(C_\bullet, \partial)$ be a chain complex of free abelian groups, then there exists a split exact sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}, G) \to \mathrm{H}^n(\mathrm{Hom}(C_\bullet, G)) \to \mathrm{Hom}(\mathrm{H}_n, G) \to 0$$

which induces isomorphisms

$$\mathrm{H}^n(\mathrm{Hom}(C_\bullet, G)) \simeq \mathrm{Hom}(\mathrm{H}_n(C_\bullet), G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(C_\bullet), G)$$

Proof

Let B_n be n-boundaries and Z_n be n-cycles, which are both free. We have exact sequences

$$0 \to B_n \to Z_n \to \mathrm{H}_n \to 0, \quad 0 \to Z_n \to C_n \to B_{n-1} \to 0.$$

This implies exact sequences

$$0 \to \mathrm{Hom}(\mathrm{H}_n, G) \to \mathrm{Hom}(Z_n, G) \to \mathrm{Hom}(B_n, G) \to \mathrm{Ext}(\mathrm{H}_n, G) \to 0$$

and the split exact sequence

$$0 \to \mathrm{Hom}(B_{n-1}, G) \to \mathrm{Hom}(C_n, G) \to \mathrm{Hom}(Z_n, G) \to 0.$$

Consider the commutative diagram with exact columns

$$\begin{tikzcd}
0 & 0 \arrow[d] & \\
\mathrm{Hom}(Z_{n-1}, G) \arrow[u] \arrow[r] & \mathrm{Hom}(B_{n-1}, G) \arrow[r] \arrow[d] & \mathrm{Ext}(\mathrm{H}_{n-1}, G) \\
\mathrm{Hom}(C_{n-1}, G) \arrow[u] \arrow[r] & \mathrm{Hom}(C_n, G) \arrow[r] \arrow[d] & \mathrm{Hom}(C_{n+1}, G) \\
\mathrm{Hom}(\mathrm{H}_n, G) \arrow[r] & \mathrm{Hom}(Z_n, G) \arrow[r] \arrow[d] & \mathrm{Hom}(B_n, G) \arrow[u] \\
 & 0 & 0 \arrow[u]
\end{tikzcd}$$

Diagram chasing shows this implies a short exact sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}, G) \to \mathrm{H}^n(\mathrm{Hom}(C_\bullet, G)) \to \mathrm{Hom}(\mathrm{H}_n, G) \to 0$$

which is also split due to the split of the middle column.

Theorem (Universal Coefficient Theorem for Cohomology)

Let G be an abelian group and X be a topological space. Then for any $n \geq 0$, there exists a split exact sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(X), G) \to \mathrm{H}^n(X; G) \to \mathrm{Hom}(\mathrm{H}_n(X), G) \to 0$$

which induces isomorphisms

$$\mathrm{H}^n(X; G) \simeq \mathrm{Hom}(\mathrm{H}_n(X), G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(X), G).$$

Proof.
Apply the previous Proposition to $C_\bullet = S_\bullet(X)$. $\square$

Universal Coefficient Theorem for homology

Definition

Let G be an abelian group. Let $A \subset X$ be a subspace. We define the relative singular chain complex with coefficient in G by

$$S_\bullet(X, A; G) := S_\bullet(X, A) \otimes_{\mathbb{Z}} G.$$

Its homology is called the relative singular homology with coefficient in G, denoted by $\mathrm{H}_\bullet(X, A; G)$. When $A = \emptyset$, we simply get the singular homology $\mathrm{H}_\bullet(X; G)$.

Similar long exact sequence for relative singular homologies follows from the short exact sequence

$$0 \to S_\bullet(A; G) \to S_\bullet(X; G) \to S_\bullet(X, A; G) \to 0.$$

Theorem (Universal Coefficient Theorem for homology)

Let G be an abelian group and X be a topological space. Then for any $n \geq 0$, there exists a split exact sequence

$$0 \to \mathrm{H}_n(X) \otimes G \to \mathrm{H}_n(X; G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(X), G) \to 0$$

which induces isomorphisms

$$\mathrm{H}_n(X; G) \simeq (\mathrm{H}_n(X) \otimes G) \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(X), G).$$